The String Topology Loop Coproduct and Cohomology Operations

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The purpose of this note is to explore the relationship between cohomology operations in a generalized cohomology theory h^* with products and a string topology loop coproduct

$$c : h^*(LM) \to h^*(LM \times LM)$$

dual to the Chas–Sullivan loop product [2]. The exact definition of c will be given below. In more detail, we are interested in giving a description for the failure of commutativity in the diagram

$$\begin{array}{ccc} h^*(LM) & \xrightarrow{c} & h^*(LM \times LM) \\ \alpha\big\downarrow & & \big\downarrow\alpha \\ h^*(LM) & \xrightarrow{c} & h^*(LM \times LM) \end{array} \tag{1}$$

when α is a cohomology operation of h^*, and will obtain a satisfactory result, phrased in terms of an exotic module structure on $h^*(LM)$ defined in terms of a characteristic class arising from α, in the case where the operation α preserves addition and multiplication. Important examples of such operations include the total Steenrod square

$$Sq = 1 + Sq^1 + Sq^2 + \cdots$$

in ordinary $\mathbf{Z}_2$-cohomology and the Adams operations

$$\psi^k : K(X) \to K(X)$$

in K-theory. In the case of the the total Steenrod square, our result parallels an earlier one by Gruher and Salvatore, who in [5] described the interaction

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of the Steenrod squaring operations with another string topology product in terms of Stiefel–Whitney classes.

The author expects the results in this note to appear as part of his PhD thesis, to be written under the direction of Professor R. L. Cohen at Stanford University. In future work, the author plans to explore the relationship between cohomology operations and other products related to string topology, such as the fusion product in twisted K-theory [4].

We will now define the the loop coproduct. Suppose M^d is a closed oriented manifold, let LM be the space of smooth loops in M, and let LM^{TM} denote the Thom space of the bundle ev^*TM, where TM stands for the tangent bundle of M and

$$ev : LM \to M$$

is evaluation at $1 \in S^1$. Then according to Cohen and Jones [3], the Chas–Sullivan loop product

$$H_*(LM) \otimes H_*(LM) \to H_{*-d}(LM)$$

arises from a certain map of spaces

$$\mu : LM \times LM \to LM^{TM}$$

and the Thom isomorphism

$$\tilde{H}_*(LM^{TM}) \xrightarrow{\approx} H_{*-d}(LM).$$

With the above interpretation of the loop product as motivation, we define the *loop coproduct* of M in a generalized cohomology theory h^* with products to be the composition

$$c : h^*(LM) \xrightarrow[\approx]{Thom} \tilde{h}^*(LM^{TM}) \xrightarrow{\mu^*} h^*(LM \times LM).$$

Of course, to ensure the existence of the required Thom isomorphism, we need to assume that our manifold M has been given an orientation with respect to h^*, and we will henceforth do so. A natural alternative way to define the loop coproduct would be to make use of the ring spectrum structure of LM^{-TM} described by Cohen and Jones and take the loop coproduct to be the composition

$$\begin{aligned} h^*(LM) &\xrightarrow[\approx]{Thom} \tilde{h}^*(LM^{-TM}) \to \tilde{h}^*(LM^{-TM} \wedge LM^{-TM}) \\ &= h^*(LM \times LM^{-TM \times -TM}) \xrightarrow[\approx]{Thom^{-1}} h^*(LM \times LM). \end{aligned}$$

It is easily checked that the two definitions agree.

Suppose now α is a cohomology operation of h^* preserving sums and products, so that

$$\alpha(x + y) = \alpha(x) + \alpha(y)$$

and

$$\alpha(xy) = \alpha(x)\alpha(y)$$

for all $x, y \in h^*(X)$ and for any space X. We would like to describe the failure of the diagram (1) to commute. Since the operation α preserves sums, it extends to give an operation (also denoted by α) on the reduced cohomology groups

$$\tilde{h}^*(X) = \mathrm{Ker}\,(h^*(X) \to h^*(pt))$$

as well as the unreduced ones. By naturality of α with respect to maps of spaces, the right hand square in

$$\begin{tikzcd} h^*(LM) \arrow[r, "Thom", "\approx"'] \arrow[d, "\alpha"'] & \tilde{h}^*(LM^{TM}) \arrow[r, "\mu^*"] \arrow[d, "\alpha"] & h^*(LM \times LM) \arrow[d, "\alpha"] \\ h^*(LM) \arrow[r, "Thom", "\approx"'] & \tilde{h}^*(LM^{TM}) \arrow[r, "\mu^*"] & h^*(LM \times LM) \end{tikzcd} \tag{2}$$

commutes, whence we see that any non-commutativity in the diagram (1) arises from the non-commutativity of the left hand square in (2).

The above observations imply that we should study the failure of α to commute with the Thom isomorphism. To do this in the proper generality, let X be a space, and let ξ be a vector bundle over X equipped with a Thom class $u_\xi \in \tilde{h}^*(X^\xi)$. Then by the Thom isomorphism theorem, we have

$$\alpha(u_\xi) = \rho_\alpha(\xi) \cdot u_\xi \in \tilde{h}^*(X^\xi)$$

for some unique class $\rho_\alpha(\xi) \in h^*(X)$. From the assumption that α preserves products, it now follows easily that for any $a \in h^*(X)$ we have

$$\alpha(a \cdot u_\xi) = \alpha(a) \cdot \alpha(u_\xi) = \alpha(a) \cdot (\rho_\alpha(\xi) \cdot u_\xi) = (\alpha(a)\rho_\alpha(\xi)) \cdot u_\xi, \tag{3}$$

whence we see that in a sense the class $\rho_\alpha(\xi)$ completely describes the failure of the square

$$\begin{tikzcd} h^*(X) \arrow[r, "Thom", "\approx"'] \arrow[d, "\alpha"'] & \tilde{h}^*(X^\xi) \arrow[d, "\alpha"] \\ h^*(X) \arrow[r, "Thom", "\approx"'] & \tilde{h}^*(X^\xi) \end{tikzcd}$$

to commute. We find it convenient to express this observation in the following form.

Proposition 1. *Let* $\mathbf{Z}[\underline{\alpha}]$ *be a polynomial ring over a single variable* $\underline{\alpha}$*, and let* $\underline{\alpha}$ *act on* $h^*(X)$ *by*

$$\underline{\alpha} \cdot x = \alpha(x)\rho_\alpha(\xi) \quad \textit{for } x \in h^*(X)$$

and on $\tilde{h}^*(X^\xi)$ *by*

$$\underline{\alpha} \cdot x = \alpha(x) \quad \textit{for } x \in \tilde{h}^*(X^\xi). \tag{4}$$

Then the Thom isomorphism map

$$h^*(X) \xrightarrow{\cdot u_\xi} \tilde{h}^*(X^\xi)$$

becomes an isomorphism of $\mathbf{Z}[\underline{\alpha}]$*-modules.*

Proof. The statement is essentially just a reformulation of (3). Notice that the assumption that α preserves sums is needed to guarantee that the given definitions make $h^*(X)$ and $\tilde{h}(X^\xi)$ into $\mathbf{Z}[\underline{\alpha}]$-modules. □

Since the $\mathbf{Z}[\underline{\alpha}]$-module structure on $h^*(Y)$ obtained by letting $\underline{\alpha}$ act as α as in (4) is clearly natural with respect to maps induced by maps of spaces, we obtain the following.

Theorem 2. *Let* $\underline{\alpha}$ *act on* $h^*(LM)$ *by*

$$\underline{\alpha} \cdot x = \alpha(x)\rho_\alpha(ev^*TM) \quad \textit{for } x \in h^*(LM)$$

and on $h^*(LM \times LM)$ *by*

$$\underline{\alpha} \cdot x = \alpha(x) \quad \textit{for } x \in h^*(LM \times LM).$$

Then the loop coproduct

$$c : h^*(LM) \to h^*(LM \times LM)$$

is a map of $\mathbf{Z}[\underline{\alpha}]$*-modules.* □

In view of Proposition 1 and Theorem 2, we should try to understand the classes $\rho_\alpha(\xi)$. The following proposition summarizes their basic properties.

Proposition 3. *The map associating to an* h^**-oriented vector bundle* ξ *over a space* X *the class* $\rho_\alpha(\xi) \in h^*(X)$ *has the following properties:*

1. *The class $\rho_\alpha(\xi) \in h^*(X)$ depends only on the isomorphism class of ξ as an h^*-oriented vector bundle over X;*

2. *$\rho_\alpha(f^*\xi) = f^*\rho_\alpha(\xi) \in h^*(Y)$ when f is a map $X \to Y$;*

3. *Given h^*-oriented vector bundles ξ over X and ζ over Y with homogeneous Thom classes u_ξ and u_ζ, we have*

$$\big[\rho_\alpha(\xi \times \zeta)\big]_k = \sum_{i+j=k} (-1)^{j|u_\xi|} \big[\rho_\alpha(\xi)\big]_i \times \big[\rho_\alpha(\zeta)\big]_j \in h^k(X \times Y)$$

where $[-]_k$ denotes the degree k part. In particular,

$$\rho_\alpha(\xi \times \zeta) = \rho_\alpha(\xi) \times \rho_\alpha(\zeta) \in h^*(X \times Y)$$

if $h^*(X \times Y)$ consists of elements of order 2, if the degree of u_ξ is even or if $\rho_\alpha(\zeta)$ is concentrated in even degrees.

Proof. Parts 1 and 2 are trivial, and part 3 follows from the computation

$$\begin{aligned}
\big[\rho_\alpha(\xi \times \zeta)\big]_k \cdot (u_\xi \wedge u_\zeta) &= \big[\alpha(u_\xi \wedge u_\zeta)\big]_{k+|u_\xi|+|u_\zeta|} \\
&= \big[\alpha(u_\xi) \wedge \alpha(u_\zeta)\big]_{k+|u_\xi|+|u_\zeta|} \\
&= \sum_{i+j=k} \big(\big[\rho_\alpha(\xi)\big]_i \cdot u_\xi\big) \wedge \big(\big[\rho_\alpha(\zeta)\big]_j \cdot u_\zeta\big) \\
&= \sum_{i+j=k} (-1)^{j|u_\xi|} \big(\big[\rho_\alpha(\xi)\big]_i \times \big[\rho_\alpha(\zeta)\big]_j\big) \cdot (u_\xi \wedge u_\zeta). \square
\end{aligned}$$

Notice that parts 1 and 2 of the preceding proposition together state that ρ_α is a characteristic class of h^*-oriented vector bundles, and that 2 and 3 combine to prove the formula

$$\big[\rho_\alpha(\xi \oplus \zeta)\big]_k = \sum_{i+j=k} (-1)^{j|u_\xi|} \big[\rho_\alpha(\xi)\big]_i \big[\rho_\alpha(\zeta)\big]_j \in h^k(X) \tag{5}$$

when ξ and ζ are h^*-oriented vector bundles over X (with homogeneous Thom classes). Also observe that in the case where h^* is ordinary cohomology with $\mathbf{Z}_2$-coefficients and α is the total Steenrod square

$$Sq = 1 + Sq^1 + Sq^2 + \cdots,$$

the class $\rho_\alpha(\xi)$ is simply the total Stiefel–Whitney class $w(\xi) \in H^*(X; \mathbf{Z}_2)$. In the case where h^* is complex K-theory and α is the k-th Adams operation

ψ^k, the class $\rho_\alpha(\xi)$ is the cannibalistic characteristic class $\rho_k(\xi) \in K(X)$ considered by Adams [1] (whence our notation ρ_α). The following well-known result aids the computation of the classes ρ_α in this case; together with the splitting principle of complex vector bundles and the sum formula (5), it in principle completely determines the classes $\rho_k(\xi) \in K(X)$ for complex vector bundles ξ.

Proposition 4. *Suppose λ is a complex line bundle over a space X. Then*

$$\rho_k(\lambda) = 1 + \lambda + \cdots + \lambda^{k-1} \in K(X).$$

Proof. It is enough to consider the case where λ is the universal line bundle η over $\mathbf{C}P^\infty$. However, in this case the Thom space $(\mathbf{C}P^\infty)^\eta$ is homeomorphic to $\mathbf{C}P^\infty$, with the Thom class corresponding to the class $\eta - 1 \in \tilde{K}(\mathbf{C}P^\infty)$ and the module structure of $\tilde{K}((\mathbf{C}P^\infty)^\eta)$ over $K(\mathbf{C}P^\infty)$ corresponding to the usual module structure of $\tilde{K}(\mathbf{C}P^\infty)$ over $K(\mathbf{C}P^\infty)$. Now the claim follows from the computation

$$\psi^k(\eta - 1) = \eta^k - 1 = (1 + \eta + \cdots + \eta^{k-1})(\eta - 1). \qquad \square$$

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